Fundamentals of Communications Engineering

Department of Communications Engineering, College of Engineering, University of Diyala, 2016-2017

Class: Second Year
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Room: Comm-02

Lecture: 09

Fourier Transform of Periodic Signals

The Fourier Series of a periodic signal is

$$x(t) = \sum_{n=-\infty}^{n=\infty} X_n e^{j2\pi n f_0 t}$$

where $X_n = \frac{1}{T} \int_T x(t) e^{-jn2\pi f_0 t} dt$

Thus $x(t) = \cdots X_{-1} e^{-j2\pi f_0 t} + X_0 + X_1 e^{j2\pi f_0 t} + X_2 e^{j2\pi f_0 t} + \cdots$

By using Linearity property of Fourier transform, then

$$X(f) = FT\left\{\sum_{n=-\infty}^{\infty} \underset{\text{these are constants}}{X_n} e^{j2\pi n f_0 t}\right\} = \sum_{n=-\infty}^{\infty} X_n \underbrace{FT\left\{e^{j2\pi n f_0 t}\right\}}_{\delta(f-nf_0)}$$

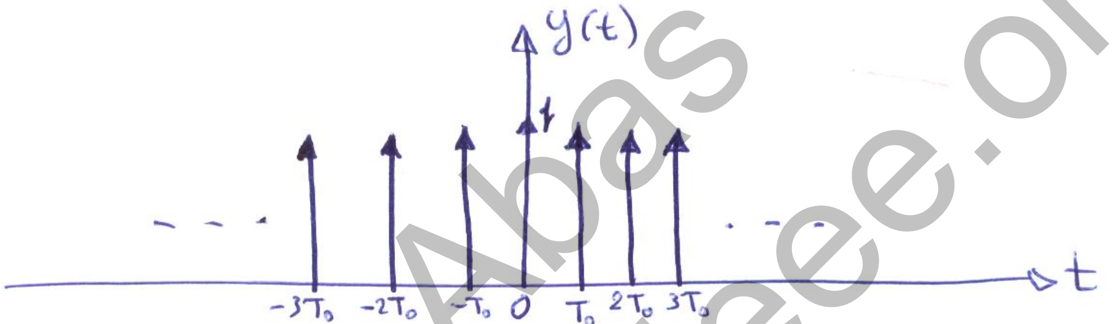
$$\therefore X(f) = \sum_{n=-\infty}^{\infty} X_n \delta(f-nf_0)$$

Fourier Transform of Impulse Train

A train of impulses can be written as

$$\delta_T(t) = y(t) = \sum_{m=-\infty}^{\infty} \delta(t - mT_0)$$

where T_0 is the fundamental period.



Since $y(t)$ is periodic, then it has a Fourier series with coefficients given as:

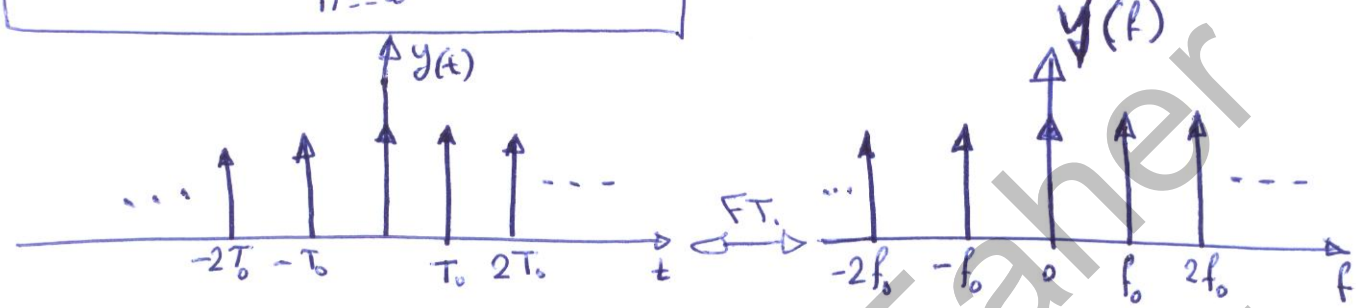
$$D_n = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} \delta(t)\, e^{-j2\pi f_0 n t}\, dt = \frac{1}{T_0}$$

$$D_n = \frac{1}{T_0} = f_0 \text{ for any } n$$

Then the Fourier transform of $y(t)$ is

$$Y(f) = f_0 \sum_{n=-\infty}^{\infty} F.T.\left\{ e^{j2\pi n f_0 t} \right\} = f_0 \sum_{n=-\infty}^{\infty} \delta(f - nf_0)$$

$$\boxed{Y(f) = f_0 \sum_{n=-\infty}^{\infty} \delta(f - nf_0)}$$



∴ In general

$$\boxed{\sum_{k=-\infty}^{\infty} \delta(t - kT_0) \xleftrightarrow{FT} f_0 \sum_{n=-\infty}^{\infty} \delta(f - nf_0)}$$

impulse train in time-domain $\xleftrightarrow{FT}$ impulse train in frequency domain

EX.1 Find the Fourier transform of a periodic signal of pulse $p(t)$ of period T_s using the convolution theorem.

Solution The periodic pulse signal can be expressed as

$$x(t) = \left[\sum_{n=-\infty}^{\infty} \delta(t - nT_0)\right] \circledast p(t) = \sum_{n=-\infty}^{\infty} p(t - nT_0)$$

where $p(t)$ represents one period of $x(t)$, with period T_0.

From the convolution theorem:

$$X(f) = FT\left\{\left[\sum_{n=-\infty}^{\infty} \delta(t - nT_0) \circledast p(t)\right]\right\}$$

$$= FT\left\{\sum_{n=-\infty}^{\infty} \delta(t - nT_0)\right\} P(f)$$

$$= f_0 P(f) \sum_{k=-\infty}^{\infty} \delta(f - kf_0)$$

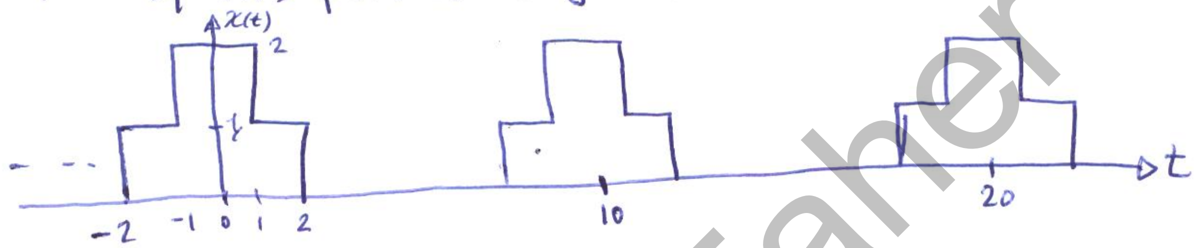
$$= f_0 \sum_{k=-\infty}^{\infty} P(kf_0)\, \delta(f - kf_0)$$

In general:

$$\boxed{\sum_{n=-\infty}^{\infty} p(t - nT_0) \xrightarrow{FT} \sum_{k=-\infty}^{\infty} f_0 P(kf_0)\, \delta(f - kf_0)}$$

EX. 2 Let $p(t) = \Pi\left(\frac{t}{2}\right) + \Pi\left(\frac{t}{4}\right)$ periodic pulse train having period $T_0 = 10$ seconds. Find the Fourier transform of this periodic signal $x(t)$.

Solution



we know $\Pi\left(\frac{t}{\tau}\right) \xleftrightarrow{FT} \tau \operatorname{sinc}(f\tau)$

$$\therefore \quad P(f) = 2\operatorname{sinc}(2f) + 4\operatorname{sinc}(4f)$$

since $p(t)$ is a periodic pulse train with period $T_0 = 10\,s \longrightarrow nf_0 = \frac{n}{10}$

$$\therefore \quad x(t) = \sum_{n=-\infty}^{\infty} p(t - nT_0)$$

then

$$X(f) = \frac{1}{10}\sum_{k=-\infty}^{\infty}\left[2\operatorname{sinc}\left(\frac{k}{5}\right) + 4\operatorname{sinc}\left(\frac{2k}{5}\right)\right]\delta\left(f - \frac{k}{10}\right)$$